Fractions and the funky cookie

A mathematics specialist has great success using a pattern-block configuration to help a small group of fifth graders understand that fractional parts of a whole unit must be equal in size. That's just the way the funky cookie crumbles.

By Aimee J. Ellington and Joy W. Whitenack

Mathematics specialists play a significant, meaningful, and integral role in supporting elementary school teachers in our mathematically connected world. These teacher leaders (coaches, resource teachers, math leads, and so on) work closely with all members of the school community—students, teachers, and administrators—and have a range of responsibilities that span working with individual teachers to working with groups of teachers and their students several times each week and sometimes daily. We highlight one of many responsibilities that a particular mathematics specialist, Ms. Sneider, has on any given day.

Sneider was working with a group of fifth-grade students who did not seem to understand that the like fractional parts that compose a whole item must be of equal size. By posing the funky cookie task (see **fig. 1**), she hoped to start a conversation about this important concept with her students, probe their thinking, and increase their understanding of fractions. The funky cookie example gives a snapshot of the types of instructional decisions that mathematics specialists might make in their daily work. The example also highlights the rich mathematical content that specialists must draw on when working with students. Sneider used her mathematical understanding to make important decisions that in turn furnished additional learning opportunities for her students.

At the outset, Sneider had not intended to address the equal-size concept. Although she had a planned activity, she made a spontaneous decision to change it. Listening to her students' ideas, she realized that she needed to address a misconception they had about fractions. The extent to which Sneider had to be flexible is reminiscent of Simon's classic example (1995, p. 133) of how a teacher may need to instruct for conceptual understanding: "Although the teacher creates an initial goal and plan for instruction, it generally must be modified many times (perhaps continually)."

Math specialists must be extremely flexible, think quickly on their feet, and make moment-to-moment decisions with multiple students at different grade levels throughout a school day.

Virginia's math specialists

A movement is currently underway in many states across the country (e.g., Oregon and Nebraska) to discuss the roles and the best placements for math specialists. In Virginia, their primary role is that of a teacher-leader who works closely with other teachers in a school to help strengthen mathematical content knowledge and enhance pedagogical practices. To this end, a specialist engages in a variety of

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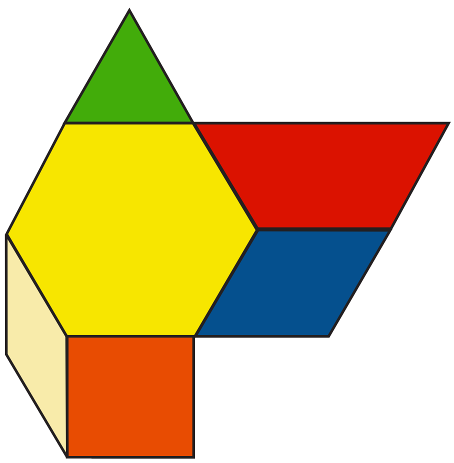
activities that range from co-planning and co-teaching with individual teachers to designing and implementing benchmark assessments for all grades in the school. Math specialists are not charged with working with individual students, but at the request of a classroom teacher (or principal), they occasionally work with small groups of students to help them better understand a particular concept. Such was the case with Sneider.

A working definition of a mathematics specialist and a detailed description of the school role he or she takes in the state of Virginia were constructed under the leadership of the Virginia Mathematics and Science Coalition (VMSC 2009). Interested readers can find more details by accessing the VAMSC Web site at http://www.vamsc.org. As part of the statewide initiative, several higher education institutions collaboratively designed a three-year program to instruct elementary school teachers who have at least three years of classroom experience and a strong interest in math. The core of the program consists of three education leadership courses designed to help specialists navigate their unique roles as coaches and leaders in their schools as well as five math courses that cover the elementary school content strands outlined by the National Council of Teachers of Mathematics (NCTM 2000) and the Conference Board of the Mathematical Sciences (CBMS 2001). One math course focuses entirely on rational numbers and proportional reasoning.

FIGURE 1

The math specialist's decision to introduce the funky cookie, which uses all six available pattern-block shapes, completely redirected the lesson she had planned.



The rational numbers course

An aim of Virginia's specialist program is to offer opportunities for participants to develop a deeper understanding of the mathematics content covered in grades K–5. Participants engage in problem-solving activities using drawings, manipulatives (pattern blocks, multilink cubes, etc.), and other approaches to model and explore important underlying mathematical ideas for various procedures and algorithms as well as other elementary school curriculum concepts. Class members often work in small groups before engaging in whole-group discussions about how to solve particular problems and the strategies they used. They also spend time exploring and analyzing children's work samples to deepen their understanding of the mathematics underlying children's methods.

Many of these activities are from materials designed for exploring fundamental ideas in the elementary school curriculum (e.g., Fosnot and Dolk 2002; Lamon 2005). Consider, for example, the activity in **figure 2**, similar to those found in Lamon (2005). Such activities give participants opportunities to grapple with proportional relationships and part-whole ideas. How might participants solve the problem using ideas about rational numbers? To answer the first part of the problem, they might separate a set of twenty-four multilink cubes (representing the stars) into four groups of equal size, which would show that each group contains six stars. They might then realize that each of these groups is one-fourth of the entire collection. To find one possible solution (multiple solution methods exist) to the second part of the question, participants might conclude that because the collection of twenty-four stars represents one-fourth of a whole set, they can use reasoning similar to what they used to answer the first part of the problem. So, if twenty-four stars make one of four equal-sized groups, the whole set contains ninety-six stars, or $24 \times 4 = 96$.

Solving problems of this type gives participants opportunities to explore ideas about rational numbers, to consider such key ideas as sets (or groups) of equal size, and to recognize the important relationship among equal parts and, more generally, the relationship between these equal parts and the whole. Notice that for both parts of the problem in **figure 2**, participants must use the concept of equal-sized groups to analyze the relationship between one-fourth and the "whole" collection. The funky cookie example addresses similar ideas. To determine the exact size of each pattern block in it, the mathematics specialist and her students had to first define the unit whole.

FIGURE 2

Participants consider proportional relationships and part-whole concepts in this sample rational numbers activity.

If a whole set is made up of the stars in the box below, how could you represent 1/4 of the stars? On the other hand, if the stars in the box represent 1/4 of a set, how many stars are in the whole set?

Background

Sneider is a full-time mathematics specialist in her school. At the time of this session with fifth graders, she had completed two years of the three-year program, including the specialists' rational numbers course during the previous summer. During Sneider's final year in the program, the authors had regular opportunities to visit her classroom, observe her work with students and teachers, and interview her before and after the observations.

Fifth-grade teachers had asked Sneider to work with a small group of children in advance of a benchmark quarterly assessment that students take to prepare for the statewide math test. Sneider's overarching goal for these sessions was to help students understand—or refine their understanding of—fractions as numbers that they could represent and reason about using the four operations. She wanted to build their understanding about concepts and strategies that they could use to solve problems—even problems of a traditional sort.

The small-group session

Before meeting with the students, Sneider planned an activity to help them understand procedures for converting improper fractions to mixed numbers. To accomplish her goal, she decided to use pattern blocks—manipulatives with which the children were well familiar.

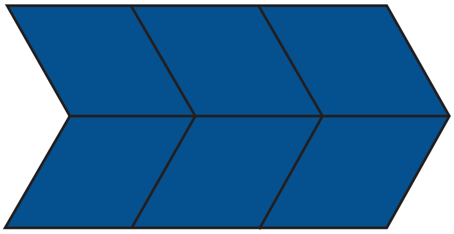
Sneider began the session by establishing the yellow hexagon as the unit. You may recall that a standard set of pattern blocks has six geometric shapes—green equilateral triangles, blue rhombuses, tan rhombuses, orange squares, red trapezoids, and yellow hexagons. Each shape has sides that are one inch in length except for the longest side of the red trapezoid (see **fig. 1**).

As a warm-up activity, each student assembled a hexagon from a combination of other pattern blocks. The three boys who participated in this session each chose a different color of pattern block to work with (red, green, or blue). The first student built a hexagon from six green triangles; the second built a hexagon from three blue rhombuses; and the third built a hexagon from two red trapezoids. After completing this task, students were to create pattern-block shapes that represent mixed numbers and then

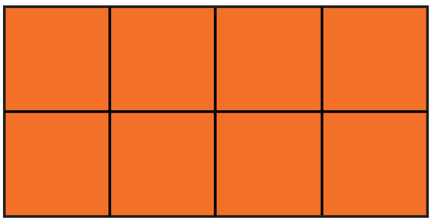
FIGURE 3

After the boys determined that the funky cookie could not be shared fairly, each student used a single shape to build a "cookie" they *could* share fairly.

(a) Student 2 used six blue rhombuses.



(b) Student 1 used eight orange squares.



discuss how they might rename these configurations using improper fractions.

After the students had built their configurations for the yellow hexagon unit, Sneider asked them to explain what one shape (e.g., one green triangle or one blue rhombus) represents when compared with the whole unit. The boy with the red trapezoids correctly answered, "One-half," and the boy with the three blue rhombuses correctly answered, "One-third." But when Sneider asked them to describe their reasoning, neither could explain why the shape represents the fraction they had named.

The other student, working with green triangles, correctly identified a green triangle as one-sixth of the hexagon and, when asked to explain, stated that the triangle is one of six parts that make the hexagon shape. On the basis of these responses, Sneider was unable to determine whether the students understood that the shapes used to make the whole had to be the same size. She sensed that she needed to probe further.

Sneider's plan changed almost immediately. To address the boys' incomplete understanding, Sneider suggested, "Let's build another whole," and proceeded to use all six available pattern-block shapes to make a configuration that she called *the funky cookie* (see **fig. 1**). Her decision to introduce the funky cookie changed the direction of the lesson completely. For the remainder of the session, the students explored the concept of equal parts as they compared those parts to a whole unit. Sneider asked them what fractional part the yellow hexagon block represents in the funky cookie. One student hesitantly stated that it is one-sixth of the whole, evidently focusing on the number of parts without considering their size.

To further explore their understanding, Sneider asked if the pieces are the same size. The boys claimed that they are not. She asked who would be the happiest if six people shared the funky cookie by splitting it into its individual pattern-block pieces. Such questions encourage students to grapple with whether the funky cookie *could* be shared fairly. The students eventually agreed that if six people share the funky cookie by each taking one of the six pattern blocks used to make the cookie, they would not receive the same amount.

During the discussion, Sneider continued to focus on the fact that the pieces in the funky cookie are not the same size. Although the yellow hexagon is not a precise fractional part of the whole cookie, she asked about its size particularly—but not so that students would offer a numerical answer. The boys had no need to calculate to determine that the hexagon is larger than the other pieces. They could easily see that the funky cookie cannot be shared fairly by simply splitting it into the six pattern-block pieces.

After this discussion, Sneider took a different approach to reinforce the same ideas. Pointing to the funky cookie, she asked, "Can you make it fair?" She also had a follow-up question ready: She asked each student to use one shape to build a "cookie" that they could share fairly with their friends. Student 3 constructed a hexagon from six green triangles; student 2 constructed another shape from six blue rhombuses (see **fig. 3a**); and student 1 built a rectangle from eight orange squares (see **fig. 3b**).

Observing, the authors were unclear about why student 1 made his rectangle with eight orange squares, but his configuration was not a problem for Sneider. As she and the students talked about the different cookies in the subsequent discussion, they referred to the number of pieces among six people and eight people interchangeably, depending on the number of pattern-block pieces used to make each cookie. The discussion seemed to flow smoothly, Sneider taking the opportunity to compare the funky cookie (see **fig. 1**) to the rectangle (see **fig. 3b**) and using the two cookies to revisit the concept of equal-sized pieces. Part of their dialogue follows. Students' comments are presented as **S1**, **S2**, and **S3**, respectively; and Sneider's as **T**.

T: What is the difference between the funky cookie and [student 1's] rectangle shape?
S1: The rectangle can be shared fairly.
T: Is the funky cookie fair? How is it different?
S2: In the rectangle, each person gets one-eighth.
T: Why is it called *one-eighth*?
S1: There are eight orange pieces.
T: What is the hexagon piece called in the funky cookie?
S3: *One-sixth.*
T: Why would you call it *one-sixth*? Does anyone agree or disagree? Think about the pieces in [student 1's] rectangle shape. Would you be happy with one orange square here? [*She picks up the orange square in the funky cookie.*]
S1: No.
T: Which group of students would be happier? [*She points to the funky cookie and the rectangle.*]
S1: The students sharing the rectangle shape.

Sneider's question about fairness proved to be an important instructional move, helping the students determine whether the yellow hexagon piece is one-sixth of the whole funky cookie and whether one orange square is one-eighth of the rectangle built of eight orange squares. Here again, this specialist continued to make decisions about how to address an important concept, using students' pattern-block configurations to develop and refine their ideas and build better understanding of fractional parts as equal-sized pieces. Through their dialogue, the boys realized that the hexagon is not one-sixth of the funky cookie and that equal fractions require equal partitions of the whole.

TABLE 1

Making calculations to find the area (in square inches) of the funky cookie and its six separate pieces involves square roots—skills beyond a typical fifth grader's. But teachers can see from this table that no two shapes are of equal area.

Shape	Exact Area	Approximate Area	% of whole
Funky Cookie	$3\sqrt{3} + \frac{3}{2}$	6.696	100.00
Triangle	$\frac{\sqrt{3}}{4}$	0.433	6.47
Square	1	1.000	14.93
Hexagon	$\frac{3\sqrt{3}}{2}$	2.598	38.80
Trapezoid	$\frac{3\sqrt{3}}{4}$	1.299	19.40
Blue Rhombus	$\frac{\sqrt{3}}{2}$	0.866	12.93
Tan Rhombus	$\frac{1}{2}$	0.500	7.47

The underlying math

Although the students could tell by looking at the funky cookie that some pieces were larger than others, they most likely could not have known for sure that each piece was a different size. Sneider may not have known the actual size of each piece, either, but she certainly could have calculated the sizes by measuring and could have added to determine the funky cookie's area. (Readers, please stop here and, if possible, make these calculations yourself before continuing.)

Table 1 gives the areas of the six pattern blocks in square inches. To determine the area of each piece, Sneider might use the fact that the side of each shape is one inch in length. The proportional relationships among the yellow, red, blue, and green shapes make the areas for these four blocks fairly easy to calculate. She might apply the Pythagorean theorem to find the height of the green triangle:

$$\frac{\sqrt{3}}{2} \text{ inches}$$

Then she might use the information to determine its area:

$$\frac{\sqrt{3}}{4} \text{ square inches}$$

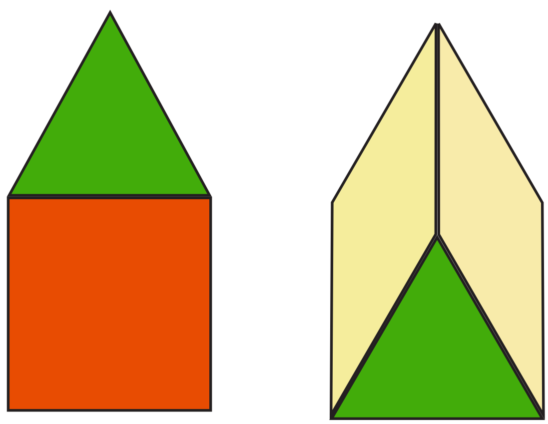
The blue rhombus, the red trapezoid, and the yellow hexagon can be generated from two, three, or six green triangles, respectively; so, Sneider could derive the area of each using the area of the green triangle. The orange square clearly has an area of one square inch.

The tan rhombus may be the most difficult calculation to make. To find its exact area, Sneider might use a protractor to first find the 150-degree measure of one of the large angles. Once she has this measure, she can use a trigonometric formula to determine the shape's height: one-half inch. She might also reach the same conclusion by comparing two tan rhombuses and one green triangle with one orange square and one green triangle (see **fig. 4**),

Because the calculations above involve square roots, this type of problem is beyond the scope of a typical fifth-grade mathematics curriculum.

FIGURE 4

The area of the tan rhombus may be the most difficult to calculate. One method is to use house shapes to show that the area of the rhombus is one-half the area of the orange square (Burns 2008).



That said, Sneider might find this information useful for her own understanding about the relationships among the shapes that the funky cookie comprises, the shapes in a standard pattern-block set. Furthermore, she may notice that when she includes the tan rhombus and the orange square in the mix, she cannot use the proportional relationships among the other four shapes to compare all the shapes to one another. She now has evidence that no two shapes are of equal area. Moreover, she may notice that the largest block, the yellow hexagon, is actually about 39 percent of the area of the funky cookie.

Final remarks

Sneider made a significant change to her original lesson plan. Working with students during the planned warm-up activity, she sensed that they understood that they needed six pieces to make the whole but did not necessarily understand that those pieces had to be equal in size. Once she realized their lack of a solid understanding of what characterizes fractional parts, she decided to replace her original lesson with several tasks in which the boys used pattern blocks to explore fractional parts.

Her decision to construct the funky cookie was a crucial point in her revised lesson. The decision to make a pattern block configuration that involved *unequal* pieces was a particularly important one, because it explicitly highlighted the students' misconception about fractional parts. By focusing a discussion around the issue of fairness, she provided an opportunity for these students to explore equal-sized parts and the important connection between equal-sized parts and fractional pieces of a whole unit.

Although she had not planned to pose these particular problems, Sneider used her understanding of key mathematical ideas to support students' reasoning about fractional parts. When she sensed students' misunderstanding, she made an important, impromptu decision; changed her plans; and built a conceptually rich activity to address the misunderstanding and advance her instructional goals. As she stated after the lesson, "Sometimes you have to do that. You have to capture the moment. I just felt it was right to do that."

Sneider credits experiences in the rational numbers course for her agility in posing the tasks that challenged students to consider some important, fundamental ideas about fractions. Her own work with fractions and proportional reasoning made her more aware of which ideas students struggle with and what she might need to address as she works with students.

The funky cookie example also highlights opportunities to delve deeply into the mathematics that children learn. Grappling with concepts gives mathematics specialist program participants opportunities to develop rich foundations from which they might draw as they work with children and teachers in their schools.

Although the funky cookie lesson occurred during a pullout session, it might just as easily have occurred as Sneider observed or co-taught a math lesson in a fellow teacher's classroom. The lesson gives us a better understanding of one of the many aspects of a math specialist's daily work and how important and complex the role is. Mathematics specialists are primarily charged to work side by side with teachers to build mathematical and pedagogical skills for effectively instructing students.

How might the funky cookie example be useful in a specialist's daily work with teachers? Sneider could share what happened during this session with the teachers in her school, either modeling the activity or making her instructional moves an explicit topic of conversation to encourage teachers to use similar methods to assess their students' understanding.

Sneider was able to draw on her understanding of teaching, leading, and mathematics to support students as they grappled with ideas about fractions. As she faces the challenge of carefully crafting activities for teachers so that they too might have similar opportunities to develop or refine their students' mathematical understanding and their own, she can draw on the thoughtful, fruitful work that she engages in daily. By hearing part of her story, we can learn from her example.

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REFERENCES

Burns, Marilyn. *The Solution to Marilyn Burns Pattern Block Puzzler*. 2008. http://teacher.scholastic.com/products/instructor/burnssoln.htm.

Conference Board of the Mathematical Sciences (CBMS). *The Mathematical Education of Teachers*. Providence, RI, and Washington, DC: American Mathematical Society and Mathematical Association of America, 2001.

Fosnot, Catherine, and Maarten Dolk. *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents*. Portsmouth, NH: Heinemann, 2002.

Lamon, Susan. *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*. Mahwah, NJ: Lawrence Erlbaum Associates, 2005.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

Simon, Martin. "Reconstructing Mathematics Pedagogy from a Constructivist's Perspective." *Journal for Research in Mathematics Education* 26 (1995): 114–45.

Virginia Mathematics and Science Coalition (VMSC). "Mathematics Specialists in Virginia." 2009. www.vamsc.org/index2.html.

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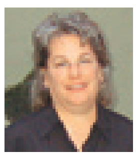

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