Supporting Middle School Mathematics Specialists' Work: A Case for Learning and Changing Teachers' Perspectives[1]

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Abstract: In this paper, we highlight one whole-class discussion that took place in a middle school mathematics *Rational Number and Proportional Reasoning* course, one of the six mathematics courses teachers take to complete our state-wide middle school mathematics specialist program. Statistical measures indicate that teachers made gains in their understanding of concepts and substantial gains in their views of teaching and preparedness. We provide a microanalysis of one of the lessons, to explain, in part, how they might have made this progress. To develop our argument, we coordinate a social analysis with an analysis of the types of specialized mathematical knowledge that teachers might have considered as they engaged in these discussions. As we will illustrate, these types of classroom discussions provided teachers opportunities to consider new visions for mathematics learning and teaching.

Keywords: Proportional Reasoning, Mathematics Specialists, Professional Development, Middle School Mathematics

Professional development initiatives that provide continuing, quality support for middle school teachers have received renewed attention in recent years. For instance, Smith, Silver and Stein (2005) stated that due to students' "lower-than-expected performance on national and international assessments" (p. *xi*) the National Science Foundation provided financial support for developers to create new middle school mathematics curricula (e.g., MathScape, Connected Mathematics Project & Mathematics in Context). that offered new innovations in teaching and learning mathematics (Reys, Reys &

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Chávez, 2004). Providing new curricula and professional development around implementing these curricula can be catalysts for teachers to further develop (or change) practices, make connections among ideas, and better support student learning (Reys, et al., 2004). However, if teachers do not develop new kinds of practices they may not be able to successfully implement innovative curricula. As Smith, Silver and Stein (2005) state with regard to implementing new middle school curricula,

> In short, new curriculum materials are unlikely to have the desired impact on student learning unless classroom instruction shifts from its current focus on routine skills and instead focuses on developing student understanding of important mathematics concepts and proficiency in solving complex problems. (p. *xi*)

Schifter and Lester (2005) mirror Smith et al.'s (2005) position. Speaking about teachers' participation in the *Developing Mathematical Ideas* programs, they state that if teachers do not "construct new visions for mathematics, mathematics learning and the mathematics classroom" (Schifter & Lester, p. 97), instructors will not be able to implement these curricula in ways that the developers intend.

Schifter and Lester's (2005) position is a useful way to frame our work in our statewide mathematics specialist program for middle school teachers. One of the aims of this work is to help teachers, when needed, to make shifts in their instructional practices so that they can effectively serve as mathematics teacher leaders, who we refer to as mathematics specialists. Our goal is to prepare middle school teachers such that once they successfully complete this program, they will be well positioned to provide ongoing, long-term, classroom-based professional development for fellow teachers in their school buildings.

Throughout the program, we know that the course instructors played a key role in helping teachers reflect more deeply about different aspects of their work (cf. Ball, Thames & Phelps, 2008). For instance, teachers reported that course instructors played a key role in helping them develop deeper understandings in the first two courses (*Numbers & Operations*; *Rational Numbers and Proportional Reasoning*) (Moffet, Fitzgerald & Smith, 2011). Additionally, teachers made statistically significant gains in their understanding of mathematics content as well as how to better teach these content ideas ($p < 0.05$) (Moffet et al., 2011). Also, they made substantial gains in their perceptions of their understanding of content and teacher preparedness. These findings have prompted us to ask the following questions: What happened during the courses that may have provided opportunities for teachers to make these kinds of shifts? What was the nature of instruction that allowed these changes to occur? How might we better understand the instructors' role in supporting the teachers' understandings of content and their perceptions of themselves as teachers of mathematics? What mathematical ideas for teaching might teachers consider as they engage in these discussions? The purpose of this paper is to unpack one of the lessons in the *Rational Numbers and Proportional Reasoning* course to understand the process by which teachers may have made these shifts in their understandings. We are particularly interested if we can identify instances during the lesson in which teachers had opportunities to consider alternative ways to reason about pedagogical and mathematical ideas. If we can identify such instances, we may gain insight into what and how they may have made these possible shifts in their perceptions and understandings of teaching and content.

To accomplish this task, we provide a microanalysis of one of the lessons in which the participants explored inverse proportions. We chose this lesson because it illustrates how the instructors and teachers established collective ways to reason about proportion problems and, as they did so, created opportunities for teachers to explore their beliefs about and commitments to teaching and learning mathematics for understanding (Shifter & Lester, 2005). Additionally, our example illustrates the some of the challenges that instructors encounter as they attempt to address teachers' more traditional views of mathematics teaching by engaging them in more innovative practices.

In the next sections, we first briefly outline our research efforts. Following this discussion, we highlight constructs that are informing our research about teachers and their work as mathematics specialists—the mathematical knowledge that they need to know to do this work (Ball, Thames & Phelps, 2008). We then analyze the lesson to understand the reasons behind the progress made by the teachers during the course. Finally we offer some comments about the importance of engaging teachers in these types of learning experiences.

Methodology Issues

In this section we outline the methods we used to analysis the classroom episode. Before doing so, we provide background about the mathematics specialist program.

Mathematics Specialist Program

The mathematics specialist program is the result of a concerted effort for over 20 years among stakeholders (university faculty, school district personnel, state professional organizations and the State Department of Education) to provide endorsement programs for K-8 mathematics specialists. Mathematics specialists are thought to have a particular set of responsibilities in their school buildings:

1. *Support* teachers through coaching, co-teaching, and modeling lessons,
2. *Translate* mathematics standards and research into classroom practice,
3. *Plan and facilitate* in-school practice-based professional development, and
4. *Work collaboratively* with administrators and staff to improve student learning.

(Virginia Mathematics & Science Coalition, n.d.)

There has been a growing interest in supporting mathematics specialists, coaches or instructional leaders in many different states. For instance, states across the country have received federal support to implement and determine the effectiveness of mathematics teacher leader programs (e.g., Nebraska's *Math in the Middle Institute Partnership*, Virginia's *Preparing Virginia Mathematics Specialists*, and Oregon's *Oregon Mathematics Leadership Institute*). These and other programs were developed in part because of the need to provide extensive, on-the-job professional development for teachers of mathematics.

At the same time, several professional documents have called for qualified mathematics specialists to be placed in schools as a resource for improving instruction (e.g., Kilpatrick, Swafford & Findell, 2001; National Council of Teachers of Mathematics (NCTM), 2000; National Mathematics Advisory Panel, 2008; National Council of Supervisors of Mathematics (NCSM), 2008). The NCSM (2008) report is particularly timely in that it provides a framework for the content that mathematics teacher leaders might need to successfully support teachers' daily work.

In our program, teachers are slated to work as mathematics specialists in their districts after they successfully complete a multi-year, 36-39 credits, Masters degree program in mathematics and mathematics education leadership. The program is composed of three 5-week summer institutes that include six mathematics courses: *Numbers and Operations, Algebra and Functions, Algebra and Functions 2, Statistics and Probability, Rational Numbers and Proportional Reasoning, and Geometry and Measurement*. Additionally, each year, teachers enroll in one *Education Leadership* course. They also complete a research in mathematics education course that follows a blended delivery format.

Instructors used activities from different sources to address content in the mathematics courses. For instance, they adapted many of the activities in the Rational Numbers and Proportional Reasoning from the work of Smith, Stein and Silver (2005) and Lamon (2005). The Education Leadership courses were designed so that teachers would explore their own teaching, their role as a math coach and their role as a change agent in the school building and district. In the Education Leadership 1, activities addressed teaching mathematics for understanding, issues that align with reform recommendations. For instance, teachers examined the NCTM (2000) documents and Stein, Smith, Henningsen, and Silver's (2000) work on cognitively demanding tasks. In Education Leadership II & III, teachers learned about coaching and working as a mathematics leader in the school context, respectively. Additionally, these courses were not taught in isolation, per se. When possible, instructors planned instruction so that Education Leadership activities aligned with content addressed in the mathematics courses.

The required mathematics courses address content that is not only covered in the middle school curriculum, but also content that requires teachers to use multiple representations, analyze the work of students, and make connections between procedures and the underlying mathematical ideas. Thus, teachers have a range of experiences that align with recommendations made by NCTM (2000) and The National Mathematics Advisory Panel (2008).

Throughout the program, course instructors use a problem-centered approach to teach the courses (Yackel & Cobb, 1996). Using this approach, the instructor presents one or more rich problems for which teachers do not readily know the answer. Teachers need to use their understandings to make sense of and solve these problems. They usually work in pairs or small groups to solve the problems together. The key is for them to understand the strategies that they use, and, when possible, to understand the different approaches that other classmates use. Additionally, they are expected to share their methods when the class reconvenes for a whole class discussion. During these discussions, the instructor plays the important role of deciding which ideas to capitalize on and which to place on hold, in addition to which representations might be used to provide teachers opportunities to explore ideas and make connections (Yackel, 2002).

Data and Analysis. The classroom episodes that we use are taken from our classroom data corpus of the two mathematics courses that we studied (we only collected data for two of the courses). Data include observation notes of the lessons, videotape recordings of small group and whole class discussions, digital recordings of small group discussions, digital photos of participants' work during whole class discussions and participants' individual work. Additionally, after viewing each of the lessons, we transcribed lessons to conduct further microanalyses of the entire lesson. As we reviewed our observation notes, we noted that teachers continued to struggle with using pictures, diagrams or manipulatives to illustrate mathematical ideas. We had marked this particular lesson as a potentially pivotal one. Although teachers continued to have various views on if they might be able to represent and solve problems and, if so, how to actually do it, during this lesson, they reasoned sensibly about proportion ideas as they used manipulatives and diagrams. For this reason, we believe that this whole class discussion was particularly important.

To conduct a microanalysis, we engaged in a process that is similar to that of Glaser and Strauss' (1967) constant comparison method. We first viewed the videotape as we analyzed the transcript of the whole class discussion. As we watched the videotaped lesson, we identified the mathematical ideas that surfaced and clarified the different models that participants used to explain solution methods. We then reanalyzed the transcript of the lesson, line by line, and made conjectures (or refuted conjectures) about how representations emerged as participants engaged in the conversation. As we did so, we also integrated each subsequent participant's contribution to further support our conjectures about if and how the participants used these representations to explain and justify their thinking. As part of this process, we made inferences about the participants' expectations and obligations in relation to their interactions with others' contributions. Through this process, we developed a more general theme about how the participants established ways to reason mathematically using multiple representations.

Theoretical Issues

Our Assumptions

We view classrooms as social settings in which teachers and their students together establish a classroom community (e.g., Ball & Bass, 2003; Cobb & Yacel, 1996). It does not matter how we might characterize the classroom or the teachers' and their students' established ways of acting and participating that are particular to that community or classroom microculture. Together, the teacher and students constitute what counts as knowing and doing mathematics. When individuals in a social setting, such as in classrooms, agree on ways of acting and participating, we refer to these as taken-as-shared practices (e.g., Cobb & Yackel, 1996; Simon & Blume 1996). Ball and Bass (2003) refer to this notion as public knowledge. Classroom practices are said to be taken-as-shared or public if and only if they are normative, that is, they are agreed upon, and eventually taken for granted by the classroom participants. As such, classroom practices are social constructions that emerge during classroom interactions. This is not to say that individual contributions do not play an important role. Different individuals may participate in these practices in different ways given their understanding of the ideas at hand (Cobb & Yackel, 1996; Ball & Bass, 2003). Although practices are socially accomplished, individuals contribute to and participate in these practices in different ways. Further, their understandings constrain and enable how they might participate in particular practices (e.g., Whitenack & Knipping, 2003)

Background

Teachers had opportunities to solve a range of tasks that were likely different from those that they used in their own classrooms to teach proportional reasoning. Engaging in , what for them were novel activities, posed challenges for many of the teachers. They seemed to address these challenges in different ways. For instance, some teachers embraced the idea of using manipulatives to solve tasks because they began to see that their students might benefit from using manipulatives or diagrams. Others, who had worked in elementary as well as middle school classrooms, were more familiar with using manipulatives to reason about ideas or to represent their thinking. Still others had little experience with using manipulatives in their classrooms. Additionally, they struggled to use different representations to reason about and to solve tasks. So teachers had varying experiences (and views) about using manipulatives and, more generally, employing multiple representations to reason mathematically. For example, in the lesson we examine below, not all of the teachers successfully used pattern blocks to solve the inverse proportion problem.

Mathematical Knowledge for Teaching

We draw on the work of Ball and her colleagues (e.g., Ball, Lubienski & Mewborn, 2001; Ball, Hill & Bass, 2005; Ball, Thames & Phelps, 2008) to understand the kinds of mathematical knowledge that teachers must have and use when teaching mathematics for understanding. As Ball (2002) asserts, mathematical knowledge for teaching [MKT] is not simply a list of mathematical skills or content that is learned as one participates in traditional mathematics courses. It is a specified type of knowledge teachers must have to effectively teach mathematics.

Ball, Thames, & Phelps (2008) separate MKT into two domains (1) common content knowledge (CCK), mathematical content and skills used in various aspects of work and everyday life—not just in the classroom, and (2) specialized content knowledge (SCK), mathematical content and skills that particularly apply to the teaching profession. Teachers need to draw on both kinds of knowledge in their work with students. With regard to SCK, teachers need to understand the important mathematical concepts that are behind a particular procedure or how to best highlight students' drawings to focus a discussion related to those ideas. With regard to CCK, teachers also need to have a deep understanding of the mathematics that they teach.

What content knowledge do teachers need to know to understand proportional reasoning? Lamon (2005) argues that to reason proportionally, teachers need to reason multiplicatively about the relationships among two or more ratios. Consider, for instance, a problem from Lamon's (2005, p. 99) text: *If 3 pizzas serve 9 people, how many pizzas will I need to serve 108 people?* To solve this problem, the teacher might recognize that the number of people will always be three times the number of pizzas. So 108 pizzas would feed 36 people—one-third of the number of pizzas. Or the teacher could reason that since there are three pizzas for nine children, there are 30 pizzas for 90 children (there are 10 times as many pizzas and children). And she knows that 33 pizzas will feed 99 children. She then adds six more pizzas and 18 more children to arrive at the answer of 36 pizzas for 108 children. Here again the teacher is said to reason proportionally since she relates pizzas and children multiplicatively (Lamon). Additionally, one can explore different relationships among ratios. For instance, two variable quantities can relate directly, or be directly proportional, if their ratio is constant. Our example of pizzas and people above is an example of ratios that are directly proportional since each is equivalent to the same constant, ⅓ (i.e., each pizza serves three people). By way of contrast, two variable quantities are inversely proportional if their product is constant.

As we analyze the whole class discussion, we will highlight some of the specialized content knowledge that might be in the background during the discussion. We do so to illustrate how closely related specialized knowledge for teaching (e.g., how different manipuatives exploit different aspects of proportional reasoning) and the teachers' solution methods are in this particular lesson. Although it was not the instructors' intent to address specialisted knowledge for teaching explicitly during the lesson, these ideas can naturally surface as teachers reflect on their learning experiences in relation to their own teaching practice.

Using Novel Tasks

One of the challenges that the instructors had was to help teachers understand the ideas that underpin the procedures they routinely use to solve proportional problems. The instructor might use one of several approaches to meet this challenge. He might ask teachers to explain why a particular procedure works. Or he might ask what mathematical ideas surface as teachers use these procedures. Or the instructor might pose tasks that require teachers to use different representations such as manipulatives, diagrams, or pictures, to model and solve problems. This instructional strategy, using models to solve problems, seemed to be an effective way to challenge teachers' understanding and beliefs about teaching for understanding. By requiring teachers to reason about ideas using different models, teachers had opportunities to explore the important ideas that underpin the methods that they used. Teachers did not have ways to readily solve tasks using these representations—these problems were novel ones for teachers. In the lesson that we analyze within this article, teachers did not readily know how to solve an inverse proportions problem using pattern blocks or the area model. As teachers engaged in these types of activities, first working together in small groups and then reconvening in a large group to talk about ideas, they had opportunities to develop deeper understandings of different concepts.

In the next section we analyze parts of one lesson to better illustrate when and under what conditions teachers might have developed new mathematical understandings.

The Inverse Proportion Lesson

During this part of the lesson the participants discussed their solutions for the following problem: *If nine people each work 1.5 hours, how long will it take six people to do this same work?* Teacher S had previously explained that six people would need to do more of the work since there were fewer people doing the work. As the discussion ensued, Teacher C (Tchr C) and Instructor 1 (Instr 1) discussed how Teacher C used blue rhombus and green triangle pattern block shapes to solve the problem. We enter the discussion as Teacher C explained how she used pattern blocks.

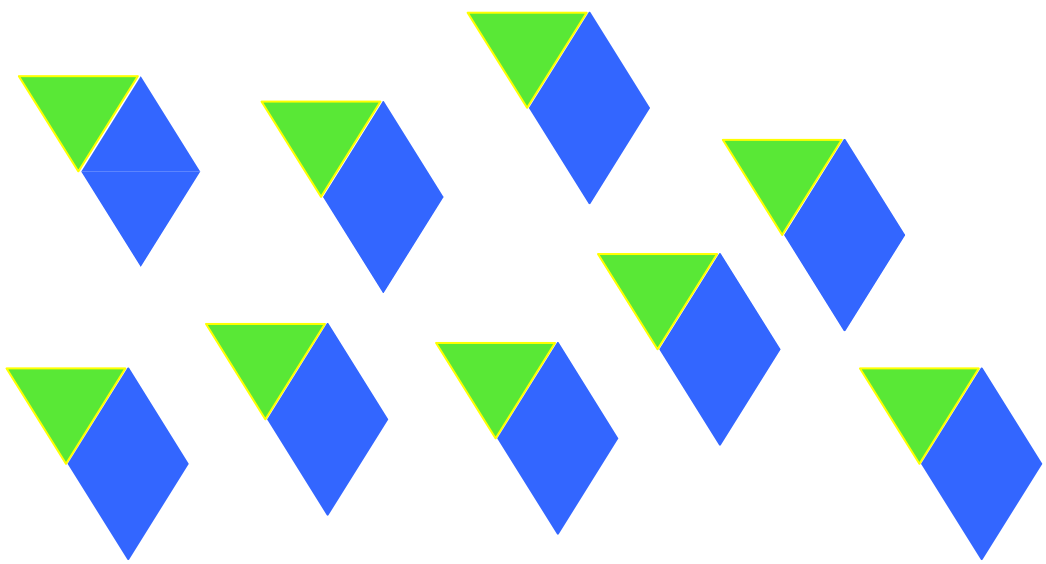

Figure 1. Instructor 1 represents Teacher C's represent of the man-hours problem.

Tchr C: I represented it with a rhombus and a triangle? So you have an hour and a half an hour. So you represent it as nine times with a blue and a green…

Instr 1: A rhombus and a triangle. [begins placing blue rhombi and green triangles to for pairs (see Figure 1)].

Tchr C: And I represented it nine times, and I thought that would show all of the time that was spent [inaudible].

As Teacher C explained how she used the blocks, Instructor 1 began making groups of blocks to represent the work that each of the nine people completed. As they engaged in this part of the discussion, teachers had the opportunity to consider how one might use the pattern blocks to solve this problem.

As the discussion continued, Teacher C explained how she would distribute the blocks to show the work that six people needed to do:

Tchr C: For me, that would represent all of the time that it took to do the job. Then I would divide that up into six piles because you only have six people. It is still going to take the same number of hours to do the job. So if you divide that into six equal piles then I should have the amount of time that it would take each person.

Instr 1: [To all the teachers] Well how would I divide nine big things and nine little things into six equal piles?

Tchrs: [Laughter and people talking over one another.] I don't know.

Notice that Teacher C made several comments that related to ideas about inverse proportions. First she explained that the nine blue-rhombus-green-triangle pairs (the number of people/hours of work) represent the total amount of work-hours. She also mentioned that if there were only six people doing the work, they would still need to complete the same number of hours of work. She also explained how she would need to determine the number of man-hours for six people. After Teacher C explained that she divided up the blocks into six piles, Instructor 1 asked the other teachers how they might divide the pattern blocks. By asking all the teachers this question, Instructor 1 invited others to engage in the discussion. As he did so, he also communicated implicitly that Teacher C's method was a viable approach for solving this proportion problem. Interestingly, in response to his question, notice too, that teachers talked over one another and some indicated that they did not know how they could divide the blocks to solve the problem.

It is at this point that Instructor 1 and Teacher C talked about how they might redistribute the blocks into six piles to solve the problem.

Instr 1: Everyone gets a green thing....So I will take out six of the blue ...[removes the 6 rhombi] trapezoids and those correspond to people working?

Tchr C: One hour.

Instr 1: One hour. And then I can take out the six of the triangles that correspond to everyone working [removes 6 green triangles]?

Tchr C: Half an hour.

Instr 1: Half an hour. That's what they were doing at the beginning when there were nine of them. That is how much work they had to do [three blue rhombi and three green triangles still presented by the document camera].

Tchr C: And now you have to trade some blues for more greens...so that you can split them all.

As Instructor 1 began distributing the six pairs of blocks, he asked what each block represented. And, each time he asked this question, Teacher C answered his question. As she did so, she and Instructor 1 continued to show how they could distribute these blocks into six equal groups. As further evidence, after distributing the six rhombi, Instructor 1 also explained that the remaining blocks (three rhombi and three triangles) were part of the man-hours they started with. Teacher C, for her part, explained that they also needed to trade rhombi for triangles so they could share all the blocks. So as he and Teacher C explained what the blocks represented at each pass, they illustrated how they might use the blocks to solve the problem.

Following this exchange, Instructor 1 and Teacher C continued to talk about how they would trade blocks and distribute the remaining three piles of rhombus-triangle pairs equally among the six groups. However, they did not find the actual values of the blocks in each of the six piles. At first we were puzzled as to why Instructor 1 and Teacher C did not actually use the pattern blocks to solve the problem. Further, it was very uncharacteristic of Instructor 1 to explain how he might use the blocks to make six equal groups. Instructor 1 usually expected teachers, not him, to explain their solution methods. So, we suspect that he never planned to solve this problem using the pattern blocks. Instead, he (and Teacher C) demonstrated the problem in order to help teachers see one possible way to use the pattern blocks to reason about this problem.

Examining the Representation

What are some of the specialized content ideas associated with using pattern blocks to solve this problem? Are there any limitations with how one can manipulate quantities when using the blocks? First, we note that the blue, yellow, red and green pattern blocks are related (1 yellow = 6 greens, 1 blue = 2 greens, and 1 red = 2 blues or 3 greens). If the blue rhombus represents 1 hour, then the green triangle represents ½ hour and together they represent 1½ hours. To represent the work of nine people, one could make nine rhombus-triangle pairs, like Instructor 1 and Teacher C did to solve the problem. Trading all the blue rhombi for green triangles, gives 27 triangles. Making six equal piles (i.e., use partitive division) yields four triangles in each pile with three leftover. So each person works 2 hours since triangles are half-hours or 4 x ½ = 2. Each person also works another 15 minutes or ¼ hour since and ½ of one green triangle is ¼ (½ x ½ = ¼). So each worker will work 2¼ hours or 2 hours, 15 minutes. Since this use of blocks requires one to mentally partition the triangles in half, one might wonder if this could be avoided by using a different block to represent the whole. Using the yellow hexagon to represent one hour, and the red trapezoid to represent ½ hour, exchanging both of these for the equivalent number of green triangles (each representing 1/6 hour) still results in needing fractional blocks to represent the solution. Thus, while pattern blocks are an appropriate way to model inversely proportional problems, the limited "denominations" of blocks can require the solution to involve mentally partitioning blocks.

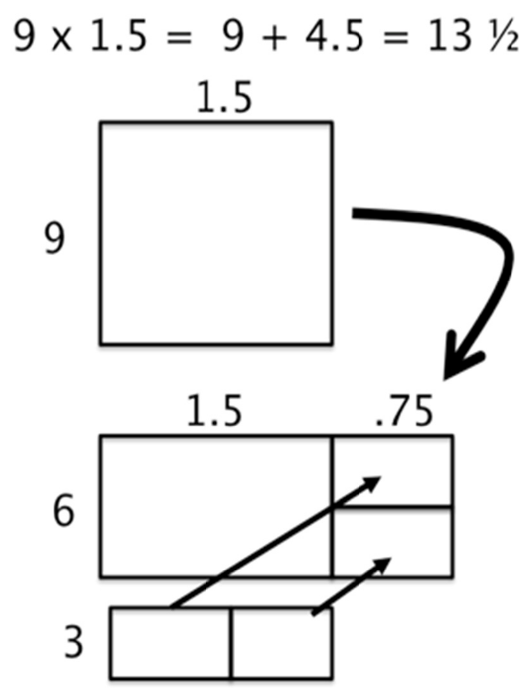

Figure 2. Teacher Leader represents the man-hours problem using rectangular regions.

Inverse Proportion Lesson—Method 2

Returning to the lesson, as the discussion ensued, Instructor 1 asked Teacher Leader, one of the other instructors, to explain his method to the class. Teacher Leader had used an area model instead of the pattern blocks to solve the problem. So as the discussion continued, Teacher Leader came to the front of the room and explained how he solved the problem using the area model. Teacher Leader explained that he first drew a 9 x 1.5 rectangle to represent 13½ man-hours. He then divided the rectangle into two smaller rectangles with dimensions, 6 x 1.5 and 3 x 1.5 (see Figure 2). Then he split the 3 x 1.5 rectangle to make two 3 x 0.75 rectangles. And he placed these two 3 x 0.75 rectangles, one on top of the other, making a new 6 x 0.75 rectangle. And finally, he adjoined this new rectangle with the 6 x 1.5 rectangle to make a 6 x 2.25 rectangle.

As the discussion continued, Teacher Leader asked the teachers if they understood how he had solved the problem. The following transcript reenters the discussion as Teacher Leader (Tchr Lead) asked the teachers if they followed his approach.

Tchr Lead: …Does everyone follow what I did?….But when I split this rectangle (3 x 1½) in half what is this value right here [points to the side that has length 0.75]? [Draws an arrow pointing to the 3 x 1½ piece now attached to the 6 x 1.5 rectangle, see Figure 2].

Tchr X: 0.75.

Tchr Lead: How did you get that?

Tchr X: Half of 1.5.

Tchr Lead: Because remember that is what I did with that area; I split that area in half so it is 0.75 [writes .75 above the 3 x 1.5 rectangle]. So now I still have the same amount of area, the same amount of work hours [moves his hand over the rectangles] that need to be done. So I kind of have to figure out what that is over here so I have 1½ hours and ¾ of an hour, so how many hours would that be?

Tchr X: 2.25.

Tchr Lead: So the men worked 2.25 or 2¼ hours [writes these two answers to the right of the new diagram].

As Teacher Leader explained his strategy, he asked the teachers if they understood how he solved the problem. Teacher X, and possibly other teachers, seemed to understand his method. As he continued to explain his diagram, notice for instance that Teacher X provided dimensions of the smaller and larger rectangles. So she and Teacher Leader, together, began to establish this second method for solving the problem.

Examining the Second Representation

Area models (continuous) offer certain advantages over pattern block models (discrete) when representing inversely proportional situations. One can continue to partition area models into smaller and smaller rectangular regions and, in the example above, evenly distribute thes13½ man-hours to each of the six people. Unlike when using the pattern blocks, one can actually rearrange these smaller partitioned pieces. One can also make different choices for how to partition the area. As in our example, Teacher Leader decomposed the rectangle with a side of length nine units into to smaller rectangles with lengths of six and three units. Additionally, the area is preserved because one is simply partitioning the given rectangle and rearranging the different parts to make a rectangle with an area of 6*n square units*.

To summarize, at this point in the lesson, both instructors have illustrated how they (and the teachers) might use two types of models to represent and ultimately solve this problem. Teacher C in our first example and Teacher X in our second example played different but important roles in substantiating that one can use these types of representations to reason about and to solve proportional problems. The instructors, for their part, asked clarifying questions and highlighted the teachers' explanations.

Interestingly, as the discussion ensued, teachers continued to question whether using these types of representations were useful. Teacher K, for instance, voiced her concern. We reenter the discussion as she commented on Teacher Leader's solution method.

Tchr K: I think…trying to explain it [this method] with… I don't understand… I'm more confused after the explanation. I mean, I know how to get the answer. I just like…the representation of it is really hard for me, for this particular problem. I can explain it. I just think that my students don't understand what I am explaining. But I feel like if I show that or the other example…they would be…and I am so confused by it, that it makes it more difficult.

Although Teacher K understood how to derive the answer, she did not understand how Teacher Leader had arrived at his answer using this representation. Furthermore, she, and possibly other teachers, did not see the relevance of using this type of representation with her students. Teacher Leader and Instructor 1 had some important decisions to make, and quickly, as to how to address Teacher K's comments.

We reenter the discussion as Teacher Leader and Instructor 1 respond to Teacher K's comments.

Tchr Lead: Are there other people that feel that way? [At least one teacher raises her hand.] Were you going to say something?

Tchr S: No. I'm just trying to figure it out.

Tchr G: In my mind, that worked very nicely because it was nine. Because you have the six [inaudible] and all that…it could have been five people. Would it work just the same?

Tchr Lead: Good question.

Instr 1: Let's try it, Teacher Leader.

Notice, in response to Teacher K's comment, Teacher Leader asked if others shared her position. In so doing, he communicated to Teacher K (and the other teachers) that he acknowledged and valued their concerns. Surprisingly, other teachers did not voice similar views. This is not to say that they did not have similar views. They simply did not voice those concerns here. Instead, in response to Teacher Leader's question, Teacher S and Teacher G commented that they were still thinking about Teacher Leader's solution method. In fact, Teacher G asked whether or not this strategy would work for other problems. Notice, too, that in response to Teacher G's question, the instructors and teachers then explored a different problem that was inversely proportional to the original problem. As the discussion continued, with a little bit of calculating, the instructors and the teachers used a similar procedure to determine that it would take five people 2.7 (i.e., 1½ + 1 + ⅕) hours to do the same work.

In retrospect, we note that Teacher K's comment was an important one. Teacher Leader's subsequent response was equally important. By asking other teachers to respond to Teacher K's comment, he and the teachers had the opportunity to explore if this method worked for other partitionings of the same rectangular region—13½. As they explored together how they might use similar methods to solve an alternate problem, they collectively established using the area model to solve these types of problems.

Mathematical Knowledge for Teaching

What are some of mathematical ideas needed to use the area model to solve inverse proportions? When one partitions a rectangular region and redistributes the area, one conserves the area of the original region. The region represents the total number of work-hours, and the dimensions of rectangular region represent the number of people and the numbers of hours each person works. One can also algebraically justify why the area is conserved. To accomplish this task, use the associative and distributive properties to generate different, equivalent expressions that represent different rectangular partitioned regions that sum to an area of 13½ square units. For example, 9 x 1 ½ = (6 + 3) x 1 ½ = (6 x 1½) + (3 x 1½). The last expression represents the new two rectangular regions with dimensions of 6 x 1½ and 3 x 1½. One can as apply the distributive property again to create another equivalent expression: 3 x 1½ = [3 x (¾+ ¾)] = (3 x ¾) + (3 x ¾) = 3 x 2 x ¾ = (3 x 2) x ¾ = 6 x ¾. This last expression represents the new rectangular region that is adjoined with 6 x 1½. So the final string of equivalent expressions is: 9 x 1½ = (6 + 3) x 1½ = (6 x 1½) + (3 x 1½) = (6 x 1½) + [3 x (0.75 + ¾)] = (6 x 1½) + (3 x 2 x ¾) = (6 x 1½) + (6 x ¾) = (6 x 2¼) = 13.5. By creating this string of equivalent expressions, we have also shown that the products of the values for each ratio are equivalent. Put another way, we have shown that the dimensions of these rectangular regions are inversely proportional since they have the same product. For the sake of brevity, we leave it to the reader to explore how they might partition this same rectangular region to show 9 x 1½ = 5 x (1½ + 1 + ⅕). (Hint: As one approach, first find the area for 5 x 1½ and 4 x 1½. Then somehow redistribute this area for 4 x 1½ to make a 5 x 1⅕ rectangle.) Finally, it is interesting to consider that there are numerous, even infinite numbers of ways to generate rectangular regions with an area of 13½ square units.

Let us now return to the ensuing discussion. Interestingly, after participants solved Teacher G's problem, the discussion returned to exploring how one might use pattern blocks to solve the inverse proportion problem. One of the teachers, Teacher M, initiated this shift in the discussion. Without prompting, she asked if she could show how she solved the problem using the pattern blocks. We reenter the discussion as Teacher M came to the front of the room and explained her thinking by sharing her work using the document camera.

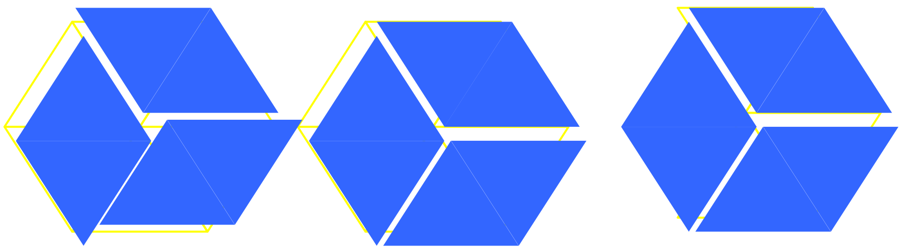

Figure 3. Teacher M shows how she used pattern blocks to solve the 9 x 1 ½ man-hours problem.

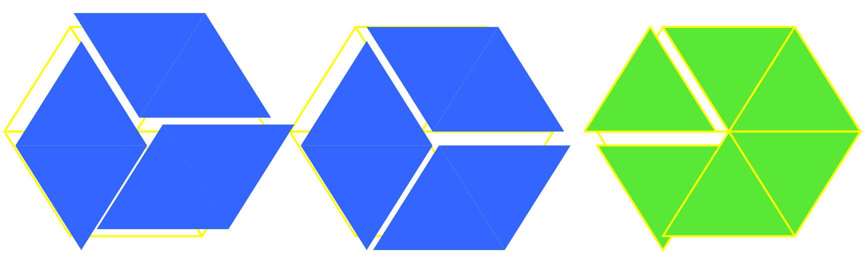

Figure 4. Teacher M trades 3 rhombi for 6 green triangles.

Tchr M: So, the three yellows [hexagon pattern blocks] were the whole. So there is the work that nine people did but we only have six people, so we have this much [removing three blue rhombi from one yellow hexagon but puts them back]…oh, and since it is an hour and a half each of these little blues are an hour and a half, but we had six people so we have this much work left to do [removes six blue rhombi from two yellow hexagons and points to the yellow hexagon, see Figure 3] so if I split that amongst six people [puts six green triangles on the yellow hexagon, see Figure 4]. Then I can see that one blue is the same as two greens. So, these are each an hour and a half [pointing at blue rhombi] so each person works an hour and a half, and also a green which is half of an hour and a half or…

Instr 1: Forty-five minutes.

Tchr M: Yeah…forty-five minutes. So then you can see, this is the same idea, they each work an hour and a half plus forty-five minutes, but less changing [than Teacher C's method] because I started with a whole. The whole was the three yellows, was all the work. Does that make sense?

Instr 1: Very nice. Does everyone understand what she just did? I think this is an illustration where one would get it right…the pattern blocks show us something, right? This solution is one that we and some children could understand. These pattern blocks aren't going to work with Teacher G's modified problem…as well. I mean, you can start off the same [relates problems by talking about pieces]…Okay. I like this. I would like to comment that this is also an example of something where we started off relatively confused with the pieces and when we ended up, we had a nice solution—a nice visual solution, medium [that] our students can understand.

It is interesting that Teacher M asked if she could show her solution method using pattern blocks. Initially, she had struggled with using the blocks. Apparently, she continued to think about the problem as the discussion ensued. She, in fact, explained in some detail why she used different blocks to solve the problem. By using this approach, she only needed to trade six triangles for three blue rhombi. She would still need to do some computing to determine what part of one hour the green triangles represented, but aside from this issue, her method, from her point of view, was more efficient—"less changing" or trading. She only needed to change out three rhombi for six green triangles before she combined one triangle with each of the blue rhombi to make six equal piles. Additionally, notice how Instructor 1 instantiated her ideas. He actually commented that her method was nice. He also mentioned that Teacher M's approach illustrated how one might use the pattern blocks to solve this problem. In fact, he suggested this was a strategy that students could understand. In so doing, he and Teacher M, continued to establish that using the pattern blocks to reason about inverse proportions was reasonable.

Mathematical Knowledge for Teaching: Comparing Solutions

Are Teacher C's and Teacher M's solution methods mathematically different? Recall in the first example, Teacher C used the triangle to represent a ½ hour, so the blue rhombus represented one hour of work. Each rhombus-triangle pair represented the work that one person completed. And the nine pairs represented the work that nine people completed for a total of 13½ man-hours. Teacher M used a different unit to show the number of hours each person worked as well as the total number of man-hours. So these two methods are different. Teacher C used the rhombus-triangle pair to represent the work of one person whereas Teacher M used only the rhombus for the same purpose. In other words, they represented to whole differently.

Interestingly Teacher M's approach seemed less cumbersome. Why? Teacher M and Teacher C may have thought about the relationships among the blocks differently. Teacher M, for instance, first represented the total number of man-hours (3 hexagons = 9 blue rhombi—1 hexagon represented the work that three people can do in 1 ½ hours). Once she had the nine pieces she only needed to trade six green triangles for the three rhombi and then redistribute these pieces. As a consequence of using the relationships among the blocks so that they better fit the problem situation, she was able to more efficiently solve the problem. By way of contrast, Teacher C represented the hours each person worked with one blue rhombus and one green triangle. So, she needed to trade blue rhombi for green triangles to redistribute the blocks.

At the close of this discussion, the instructors and teachers have contributed in part to constituting that both of these solution methods are reasonable—they can use pattern blocks or the area model to solve these types of problems. Initially using the pattern blocks to derive the solution did not seem viable to the participants. Recall that during the first part of the discussion, for instance, Teacher C and Instructor 1 did not actually solve the problem using the blocks. By the end of this conversation, when Teacher M illustrated how she could use this method, they now had established that using the pattern blocks was a viable approach. Of course, Teacher C's method was equally viable, but because they did not actually solve the problem, teachers may not have been convinced at the beginning of the lesson.

Final Comments

At the close of this discussion, the instructors and teachers began to collectively establish that these approaches were normative, reasonable ways to solve inverse proportion problems. Providing opportunities for middle school teachers to make changes in their views about using multiple representations, is a first and important step in supporting their professional learning about teaching mathematics and supporting teachers' learning. Participants played different parts in advancing discussions. For instance, Teacher M and Teacher C, along with Instructor 1, illustrated how one might use pattern blocks to solve tasks. Also, Teacher G's comment was particularly important in helping teachers consider how they might solve similar problems using Teacher Leader's approach. Additionally, Teacher K's comment about Teacher Leader's approach was important. Although she may have challenged the idea of using these types of approaches, her concerns, although acknowledged, seemed to fade into the background temporarily as participants, following Teacher G's question, continued to explore how to use the area model to solve a similar problem.

Our goal is to better understand why teachers made the progress that they did by the end of the *Rational Numbers and Proportional Reasoning* course. The pretest-posttest assessment taken by all participants in the institute revealed that all the teachers better understood the content at the end of the course but the assessment does not help us understand how and why the changes were made. Teachers demonstrated that they knew how to solve problems using more traditional paper and pencil methods. However, if they had engaged in more traditional types of activities, they would have had fewer opportunities to explore why those procedures work. And more importantly, they may not have understood the important mathematical ideas that underpin those ideas. Situations such as the ones we illustrated in this lesson, provided teachers with opportunities to explore these ideas more deeply. As teachers represented and solved problems using manipulatives, pictures and diagrams—approaches that were fairly novel for them—they had opportunities to explore the different ideas and concepts.

We suspect that other teachers may ask similar questions as they move through other courses in the mathematics specialist program. Teachers had concerns about how they might support their students' learning using similar instructional practices. As they continue in the program, it will be important for them to have opportunities to address these and other issues around teaching and mathematics. In this particular lesson, there are other questions that might arise naturally. For instance, does using the area model afford teachers more opportunities to explore proportional relationships with students? We could imagine that this issue might arise naturally as teachers continued to routinely use these types of models to reason with and about proportions. As such, teachers might explore and possibly expand their understanding of the important mathematical ideas associated with these types of proportional activities. As they do so, they may revise their views about teaching mathematics for understanding. It is critical for teachers to develop these and many other strategies in order to be effective mathematics specialists in their school buildings.

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