UNLOCKING UNDERGRADUATE PROBLEM SOLVING

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Abstract

It is difficult to find good problems for undergraduates. In this article, we explore an interesting problem that can be used in virtually any mathematics course. We then offer natural generalizations, state and prove some related results, and ultimately end with several open problems suitable for undergraduate research. Finally, we attempt to shed some light on what makes a problem *interesting*.

Introduction

The Department of Mathematics at Lynchburg College has made a concerted effort to bring serious mathematical thinking into every one of its mathematics classes. We want our students to have the opportunity to question, explore, make conjectures, and then prove those conjectures. We want them to experience the true beauty of problem solving.

We spend a great deal of our time looking for appropriate problems that can be used at many levels. We leave no stone unturned. We examine textbooks, *Math Olympiad* and similar problem books, websites, and *Car Talk* “Puzzlers” [1-3]. During these investigations, we have come across several excellent problems. We are always looking for interesting problems that satisfy the following four criteria:

1) Are easy to understand, but for which the solution is not obvious;
2) Require some experimentation and examples to make a conjecture;
3) Have some higher-level mathematics lurking in the background; and,
4) Can be easily generalized.

In this article, we will study one such problem. This problem can be found in *Mathematical Delights*, in “From the Desk of Liong-shin Hahn” as Problem 1: “A Safe Cracking Problem” [4]. The problem is as follows:

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A lock has sixteen keys arranged in a 4 x 4 array; each key is oriented either horizontally or vertically. In order to open the lock, all of the keys must be vertically oriented. When a key is switched to another position, all the keys in the same row and column automatically switch their positions, too (see diagram). (Only one key at a time can be switched.)

Show that no matter what the starting position, it is always possible to open this lock.

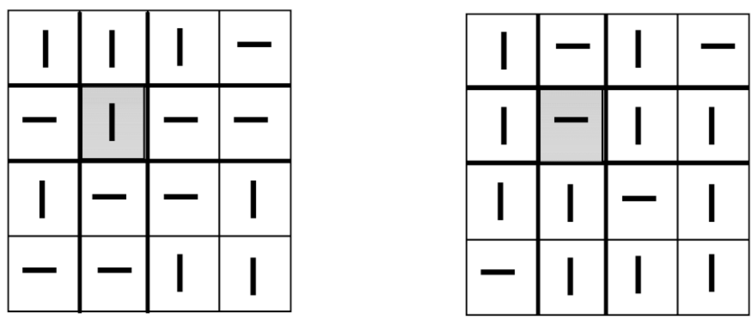

Figure 1. Original lock; lock after turning the shaded key.

This problem was given to students at many levels: to students from our liberal arts problem solving course to our upper-level students in linear algebra and experimental mathematics. All of these students agreed that the problem was easy to understand and that the solution was not at all obvious.

Problems like this force students to think differently. They spend more time practicing higher-order thinking skills than they do rummaging through their dusty old high school bag of formulas and techniques. In fact, no matter the mathematical level of the student, they immediately begin experimenting!

Results

Most mathematicians that see this problem instantly recognize the locks (in all their possible states) as elements of the vector space of 4 x 4 matrices over $\boldsymbol{Z}_2$, where an entry of 0 corresponds to a vertical key and an entry of 1 corresponds to a horizontal key. In this setting, turning a key in the (i,j) position translates to adding the matrix $A_{i,j}$ that has ones in the i^{th} row and the j^{th} column and zeros elsewhere to the matrix corresponding to the lock in question. To prove that every lock is open, one need only prove that the set of matrices $A_{i,j}$ form a basis.

Before we prove any results for lock problems such as this one, we should make a few comments. Students, even those with no background in linear algebra, quickly realize from experimentation that they can change the orientation of the key in the i^{th} row and j^{th} column while leaving all other keys in the same orientation by turning each key in the i^{th} row and the j^{th} column exactly once. This realization requires knowledge of neither linear algebra nor modular arithmetic. In fact, it turns out that students with knowledge of little other than basic parity can, with a little experimentation, come to this same conclusion. However, it is exactly this technique that can be used to prove the linear algebra version in a straightforward manner for any lock with an even number of both rows and columns:

Result 1. Any lock with an even number of rows and an even number of columns can be opened regardless of starting position.

Proof. Choose an arbitrary starting position of any m x n lock where m and n are both even. Let A be the matrix in the vector space V of m x n matrices over $\mathbf{Z}_2$ for which we choose each entry as follows: if the corresponding key in our given starting position is horizontal, the entry is 1, and if the corresponding key in our given starting position is vertical, the entry is 0.

Consider matrices of the following form in the vector space of m x n matrices over $\mathbf{Z}_2$:

$A_{k,l}$ defined by $(A_{k,l})_{i,j} = \begin{cases} 0 & \text{if } k \neq i \text{ and } l \neq j. \\ 1 & \text{if } k = i \text{ or } l = j. \end{cases}$

$A_{k,l} + A$ produces a matrix that corresponds to the lock position we would obtain by turning the key in the k^{th} row and the l^{th} column. So, if the set $S = \{A_{k,l} : 1 \leq k \leq m, 1 \leq l \leq n\}$ spans V, every matrix in V may be written as a linear combination of elements of S, in particular the zero matrix, which will allow us to conclude that any arbitrary lock may be opened.

In order to show that S spans V, it suffices to show that any arbitrary element of the standard basis of V can be written as a linear combination of elements of S. Choose an arbitrary element of the standard basis of V, $E_{i,j}$, the matrix whose $(i,j)^{\text{th}}$ entry is 1 and all other entries are zero. Clearly, we can unlock any lock using the matrices $E_{i,j}$.

We claim that $E_{i,j} = \sum_{k=1}^{m} A_{k,j} + \sum_{l=1}^{n} A_{i,l} - A_{i,j}$.

First, note that $\sum_{k=1}^{m}(A_{k,j})_{i,j} + \sum_{l=1}^{n}(A_{i,l})_{i,j} - (A_{i,j})_{i,j} = m + n - 1 \equiv 1 = (E_{i,j})_{i,j} (\text{mod}\, 2)$.

If $s \neq i$, then $\sum_{k=1}^{m}(A_{k,j})_{s,j} + \sum_{l=1}^{n}(A_{i,l})_{s,j} - (A_{i,j})_{s,j} = m + 1 - 1 = m \equiv 0 = (E_{i,j})_{s,j} (\text{mod}\, 2)$.

If $t \neq j$, then $\sum_{k=1}^{m}(A_{k,j})_{i,t} + \sum_{l=1}^{n}(A_{i,l})_{i,t} - (A_{i,j})_{i,t} = 1 + n - 1 = n \equiv 0 = (E_{i,j})_{i,t} (\text{mod}\, 2)$.

If $s \neq i$ and $t \neq j$, then $\sum_{k=1}^{m}(A_{k,j})_{s,t} + \sum_{l=1}^{n}(A_{i,l})_{s,t} - (A_{i,j})_{s,t} = 1 + 1 - 0 \equiv 0 = (E_{i,j})_{s,t} (\text{mod}\, 2)$.

Hence, S spans V and therefore any m x n lock with m and n even can be opened.

In asking our students to ask interesting questions inspired by the lock problem, a natural idea students have is to question whether the results will be the same if the number of rows and columns are changed. The result we have just shown is, in fact, one such simple extension of this problem. In asking these sorts of questions, one of our students noticed that a 3 x 3 lock had a particular position (and then a whole class of related positions) that could not be unlocked. In fact, even an introductory student can quickly discover positions for a 1 x n lock, with $n > 1$, that cannot be unlocked.

Again, as mathematicians, creating these examples is relatively simple if we treat the locks as elements of the vector space of m x n matrices over $\mathbf{Z}_2$. However, students, even those at an introductory level, can quickly create "unopenable" positions of locks of various sizes, along with most of a proof that these locks cannot be opened, even if they lack any background in linear algebra. Again, an understanding of parity is all that is required to discover these ideas. However, in a class in which Martin Gardner's famous "Mutilated Chessboard" problem is studied, students can find interesting connections between their attempts to create an unbreakable lock and that problem [5]. We will now prove a couple more results that were inspired by these students' explorations.

Result 2. If m is odd and n is even, there exist positions from which an m x n lock cannot be opened.

Proof. Examine any starting position of this m x n lock with an odd number of horizontally positioned keys. Let A be the m x n matrix over $\mathbf{Z}_2$ which corresponds to this starting position as before. Further, define the matrices $A_{k,l}$ as before. Define the function σ by:

$$\sigma(B) \equiv \sum_{i=1}^{m}\sum_{j=1}^{n}(B)_{i,j} \pmod 2.$$

An open lock which corresponds to matrix D satisfies $\sigma(D) \equiv 0 \pmod 2$, while the matrix A for our starting position above satisfies $\sigma(A) \equiv 1 \pmod 2$. Further, since $A_{k,l} + A$ produces a matrix that corresponds to the lock position we would obtain by turning a key in the k^{th} row and l^{th} column, $\sigma(B + C) \equiv \sigma(B) + \sigma(C)$ for any matrices B and C, and $\sigma(A_{k,l}) \equiv m + n - 1 \equiv 0 \pmod 2$, no sequence of key turns can open a lock in a position corresponding to matrix A.

Result 3. If m and n are both odd, not both 1, there exist positions from which an m x n lock cannot be opened.

Proof. Color the position (or cell) of the key in the i^{th} row and j^{th} column of a lock black if $i + j$ is even and white if $i + j$ is odd. The cells will appear in a checkerboard pattern, starting with a black cell in the upper left hand corner as shown in Figure 2 below.

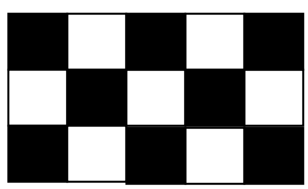

Figure 2. Colored 3 x 5 lock.

Note that each odd row and column starts and ends with a black cell, while each even row and column starts and ends with a white cell. Thus, when m (or n) $\equiv 1 \pmod 4$, since m (or n) = $4r + 1$, each odd row (or column) contains $2r + 1$ black and $2r$ white cells, and each even row (or column) $2r + 1$ white and $2r$ black cells. Also, when m (or n) $\equiv 3 \pmod 4$, m (or n) = $4r + 3$, each odd row (or column) contains $2r + 2$ black and $2r + 1$ white cells, and each even row (or column) $2r + 2$ white and $2r + 1$ black cells.

For any position of a lock, we can define an m x n matrix A over $\mathbf{Z}_2$ corresponding to that position as before. Also, define the matrices $A_{k,l}$ as before. Define the functions σ_b and σ_w which sum the entries in a matrix corresponding to the black and white cells of a lock:

$$\sigma_b(A) \equiv \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j} \pmod 2, \text{ where } x_{i,j} = \begin{cases} (A)_{i,j} & \text{if } i+j \text{ is even.} \\ 0 & \text{if } i+j \text{ is odd.} \end{cases}$$
$$\sigma_w(A) \equiv \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j} \pmod 2, \text{ where } x_{i,j} = \begin{cases} (A)_{i,j} & \text{if } i+j \text{ is odd.} \\ 0 & \text{if } i+j \text{ is even.} \end{cases}$$

An open lock which corresponds to matrix D satisfies $\sigma_b(D) \equiv \sigma_w(D) \equiv 0 \pmod 2$. Again, note that $A_{k,l} + A$ produces a matrix which corresponds to the lock position we would obtain by turning a key in the $(k,l)^{\text{th}}$ cell and $\sigma_b(B + C) \equiv \sigma_b(B) + \sigma_b(C)$ and $\sigma_w(B + C) \equiv \sigma_w(B) + \sigma_w(C)$ for any matrices B and C.

Examine the following cases.

Case 1: $m \equiv n \equiv 1 \pmod 4$

Let $m = 4r + 1$ and $n = 4s + 1$.

If k and l are even, $\sigma_w(A_{k,l}) = (2r + 1) + (2s + 1) \equiv 0$.

If k is even and l is odd, $\sigma_w(A_{k,l}) = (2r + 1) + (2s) - 1 \equiv 0$.

If k is odd and l is even, $\sigma_w(A_{k,l}) = (2r) + (2s + 1) - 1 \equiv 0$.

If k and l are odd, $\sigma_w(A_{k,l}) = (2r) + (2s) \equiv 0$.

Choose a starting position for the lock which has an odd number of white horizontal cells. Then, the corresponding matrix A to this lock satisfies $\sigma_w(A) \equiv 1 \pmod 2$. Since $\sigma_w(A_{k,l}) \equiv 0$ for any choice of k and l, no sequence of key turns can open a lock in a position corresponding to matrix A.

Case 2: $m \equiv n \equiv 3 \pmod 4$

Similar to Case 1.

Case 3: $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$

Let $m = 4r + 1$ and $n = 4s + 3$.

If k and l are even, $\sigma_b(A_{k,l}) = (2r) + (2s + 1) - 1 \equiv 0$.

If k is even and l is odd, $\sigma_b(A_{k,l}) = (2r) + (2s + 2) \equiv 0$.

If k is odd and l is even, $\sigma_b(A_{k,l}) = (2r + 1) + (2s + 1) \equiv 0$.

If k and l are odd, $\sigma_b(A_{k,l}) = (2r + 1) + (2s + 2) - 1 \equiv 0$.

Choose a starting lock position which has an odd number of black horizontal cells whose matrix A then satisfies $\sigma_b(A) \equiv 1 \pmod 2$. Again, since $\sigma_b(A_{k,l}) \equiv 0$ for all k and l, no sequence of key turns can open such a lock.

Case 4: $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$

Similar to Case 3.

Conclusion

As we mentioned earlier, this problem satisfies the four criteria that makes a problem interesting. We have proven a few results so that students might have some ideas on how to start on other generalizations. Like all interesting problems, generalizations abound! We end this paper with the following versions of lock problems:

1) Start with an m x n lock as studied above. Change the rules for which keys change when a particular key is turned. For instance, what if only the keys sharing a border with the turned key changes? What size locks can be opened?
2) Start with a locked 3-dimensional rectangular m x n x l box, each face of which is covered by two m x n, two m x l, and two n x l keys similar to those studied above. In this 3-D version, turning one key changes the positions of each key in the same row and column around the entire box.
3) Start with a locked 3-dimensional rectangular m x n x l box containing mnl cubes each containing a key. Turning any key (even those in the interior of the box) changes the position of every key sharing a horizontal or vertical plane with the key turned. Alternately, turning any key might change the position of every key sharing a horizontal or vertical row with the key turned.
4) Start with an m x n lock that corresponds to an m x n matrix over $\mathbf{Z}_r$. That is, a lock where each key has r intermediate positions between the horizontal and vertical positions.

5) Start with a locked 3-dimensional rectangular m x n x l box that works as in version 1 or version 2 above, but with keys that have r intermediate positions between horizontal and vertical positions as in version 3 above.

References

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